

RANDOM HISTORIC BEHAVIOUR

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ABSTRACT. The point with no time averages for a random dynamical system f is said to have historic behaviour. It is known that for any absolutely continuous random dynamical system of \mathcal{C}^r diffeomorphisms on a closed smooth Riemannian manifold with $r \geq 1$, if the random system is given by an independent and identically distributed (i.i.d.) random process, then the set of points with historic behaviour has zero Lebesgue measure. We prove that it does not hold if one drops the i.i.d. condition, by constructing a non-i.i.d. random dynamical system of \mathcal{C}^r diffeomorphisms having a positive Lebesgue measure set consisting of points with historic behaviour.

1. INTRODUCTION

This paper concerns random dynamical systems on a parametrised family of \mathcal{C}^r diffeomorphisms on a closed smooth Riemannian manifold M with $r \geq 1$. A *parametrised family of \mathcal{C}^r diffeomorphisms* on M is given as a differential mapping $f : B \times M \rightarrow M$ such that $f_t \equiv f(t, \cdot) : M \rightarrow M$ is a \mathcal{C}^r diffeomorphism for all $t \in B$, where B is the unit ball of a Euclidean space. For each $\bar{t} = (t_0, t_1, \dots) \in B^{\mathbb{N}_0}$, the *random orbit* $\{f_{\bar{t}}^{(n)}(x)\}_{n \geq 0}$ of $x \in M$ at \bar{t} is defined by

$$f_{\bar{t}}^{(n)}(x) = f_{t_{n-1}} \circ f_{t_{n-2}} \circ \cdots \circ f_{t_0}(x), \quad n \geq 1,$$

and $f_{\bar{t}}^{(0)}(x) = x$ (below we introduce Lebesgue probabilities on (subsets of) $B^{\mathbb{N}_0}$).

A naive expectation about the random dynamics of f is that once we impose an appropriate condition on f , the “randomness” of \bar{t} will be transmitted to the statistical properties of $\{f_{\bar{t}}^{(n)}\}_{n \geq 0}$ (and consequently, the choice of model of randomness is of great importance). A celebrated result in the direction is established by Araújo [1] in the *i.i.d. case*, which inspires the work in this paper.¹ To state his and our result precisely, we need to define *historic behaviour* for f .

Definition 1. For given $\bar{t} \in B^{\mathbb{N}_0}$, we say that the random orbit of $x \in M$ at \bar{t} has historic behaviour if there exists a continuous function $\varphi : M \rightarrow \mathbb{R}$ for which the time average

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\bar{t}}^{(j)}(x))$$

does not exist.

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¹ For another result in the direction (from the viewpoint of mixing property), refer to [8] and the references therein.

Since several statistical quantities are given as the time average of some observable φ , it is a natural question whether the set of initial points whose random orbits have historic behaviour is of positive Lebesgue measure. Particularly, in the deterministic case (i.e., in the case when $f_t = f_0$ for all $t \in B$, so that $f_{\bar{t}}^{(n)}$ is the usual n -th iteration of the single map f_0 for all $\bar{t} \in B^{\mathbb{N}_0}$, with the notation 0 for the origin of B), the problem whether the set of initial points with historic behaviour is *persistently* of positive Lebesgue measure is called *Takens' Last Problem* [10]: Very recently it was affirmatively answered by the first and third authors in [6], that will be briefly restated in Theorem 7. Furthermore, this was applied to detect a persistent class of 3-dimensional flows having a positive Lebesgue measure set consisting of points with historic behaviour in [7]. The reader is asked to see [9, 10, 3] for the background of historic behaviour in the deterministic case.

Contrastively to the deterministic case, the set of initial points with historic behaviour is of zero Lebesgue measure for most random dynamical systems in the i.i.d. setting due to Araújo [1]: Let Leb_M and Leb_B be the normalised Lebesgue measures on M and B , respectively, and let $(B^{\mathbb{N}_0}, \text{Leb}_B^{\mathbb{N}_0})$ be the product probability space of (B, Leb_B) . We say that f is *absolutely continuous* if there exists an integer $N \geq 1$ and a real number $\xi > 0$ such that for all $n \geq N$ and $x \in M$,

$$(1.2) \quad \{f_{\bar{t}}^{(n)}(x) \mid \bar{t} \in B^{\mathbb{N}_0}\} \text{ contains the ball with radius } \xi \text{ centred at } f_0^n(x),$$

$$(1.3) \quad \left(f_{(\cdot)}^{(n)}(x)\right)_* \text{Leb}_B^{\mathbb{N}_0} \text{ is absolutely continuous with respect to } \text{Leb}_M,$$

where f_0^n is the usual n -th iteration of f_0 , and $(f_{(\cdot)}^{(n)}(x))_*$ is the pushforward of measures by the measurable mapping $f_{(\cdot)}^{(n)}(x) : B^{\mathbb{N}_0} \rightarrow M$. Note that the deterministic case is excluded by assuming that f is absolutely continuous. The following theorem is an immediate consequence of [1, Theorem 1].

Theorem 2. *Suppose that f is an absolutely continuous parametrised family of \mathcal{C}^r diffeomorphisms on M with $r \geq 1$. Then for $\text{Leb}_B^{\mathbb{N}_0}$ -almost every $\bar{t} \in B^{\mathbb{N}_0}$, the set of points $x \in M$ whose random orbits at \bar{t} have historic behaviour is a zero measure set with respect to Leb_M .*

The above theorem provides a rather complete description of Lebesgue measure of the set of initial points with historic behaviour for random dynamical systems in the i.i.d. setting.² However, the random system $\{f_{\bar{t}}^{(n)}\}_{n \geq 0}$ is occasionally a model of a perturbed system under noise consisting of unknown parameters, and the noise may have autocorrelation. Hence, in order to further understand how the noise is transmitted to the random dynamics, some investigation of Lebesgue measure of the set of points with historic behaviour under non-i.i.d. noise is naturally needed. As one of the simplest non-i.i.d. noise, we here consider *noise generated by a measurable mapping*. Let $\theta : B \rightarrow B$ be a measurable mapping. We let $\{(t, \theta t, \theta^2 t, \dots) \mid t \in D\}$ be denoted by D^θ for each Borel set $D \subset B$ with the notation $\theta t = \theta(t)$, and define

² It is not difficult to see that with the notation $\text{Diff}^r(M, M)$ for the space of all \mathcal{C}^r diffeomorphisms on M , $\{\bar{t} = (t_0, t_1, \dots) \mapsto f_{t_n}\}_{n \geq 0}$ is *independent and identically distributed* (abbreviated by *i.i.d.*) $\text{Diff}^r(M, M)$ -valued random variables on $(B^{\mathbb{N}_0}, \text{Leb}_B^{\mathbb{N}_0})$, i.e., each random variable has the same probability distribution as the others and all are mutually independent. Hence, we say that Theorem 2 is a result in the i.i.d. setting.

a probability measure Leb_B^θ on $B^{\mathbb{N}_0}$ by

$$\text{Leb}_B^\theta(\Gamma) = \begin{cases} \text{Leb}_B(D) & (\text{if } \Gamma = D^\theta \text{ for some Borel set } D \subset B) \\ 0 & (\text{otherwise}), \end{cases}$$

for each measurable set Γ in $B^{\mathbb{N}_0}$.³ The study of $\{f_{\bar{t}}^{(n)}\}_{n \geq 0}$ for Leb_B^θ -almost every \bar{t} is basically equivalent to considering a *random dynamical system in the sense of Arnold* [2] over the base transformation $\theta : B \rightarrow B$. (The only essential difference is that we do not assume θ to preserve a probability measure on B .) For the recent progress in random dynamical system theory and its application to analyses of real phenomenon, see, for example, [2, 4, 5] and the references therein.

Now we can provide our main theorem:

Theorem 3. *There exist an absolutely continuous parametrised family f of \mathcal{C}^r diffeomorphisms on M with $r \geq 1$ and a nonsingular measurable mapping $\theta : B \rightarrow B$ for which we can find a Leb_B^θ -positive measure set D^θ and a Leb_M -positive measure set I_0 such that the random orbit of x at \bar{t} has historic behaviour for every $(\bar{t}, x) \in D^\theta \times I_0$.*

Remark 4. The basic strategy for the proof of Theorem 3 is to construct the parametrised family f as it transmits the historic behaviour in the noise parameter space to the phase space. From the viewpoint, we can generalise the concrete family f given in Subsection 2.1 to a more general class without much difficulty: for instance, Theorem 3 remains true when f is a “small perturbation” of a single map f_0 on M with a sink (refer to (2.1), (2.2) and Figure 1; see also Example 2 in [1]). However, we illustrate the strategy through the concrete example in Subsection 2.1, in order to keep our presentation as transparent as possible.

2. PROOF OF THEOREM 3

2.1. Parametrised family. We shall construct f in Theorem 3 as a “random perturbation” of a diffeomorphism on the circle. Let \mathbb{S}^1 be the circle given by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We endow \mathbb{S}^1 with a metric $d_{\mathbb{S}^1}(\cdot, \cdot)$, where $d_{\mathbb{S}^1}(x, y)$ is the infimum of $|\tilde{x} - \tilde{y}|$ over all representatives \tilde{x}, \tilde{y} of $x, y \in \mathbb{S}^1$, respectively. Let $\pi_{\mathbb{S}^1} : \mathbb{R} \rightarrow \mathbb{S}^1$ be the canonical projection on \mathbb{S}^1 , i.e., $\pi_{\mathbb{S}^1}(\tilde{x})$ is the equivalent class of $\tilde{x} \in \mathbb{R}$. We write I_0 for $\pi_{\mathbb{S}^1}([\frac{1}{4}, \frac{3}{4}])$. Let f_0 be a \mathcal{C}^r diffeomorphism on \mathbb{S}^1 such that

$$(2.1) \quad f_0(x) = \frac{1}{2}x + \frac{1}{4} \pmod{1}, \quad x \in I_0,$$

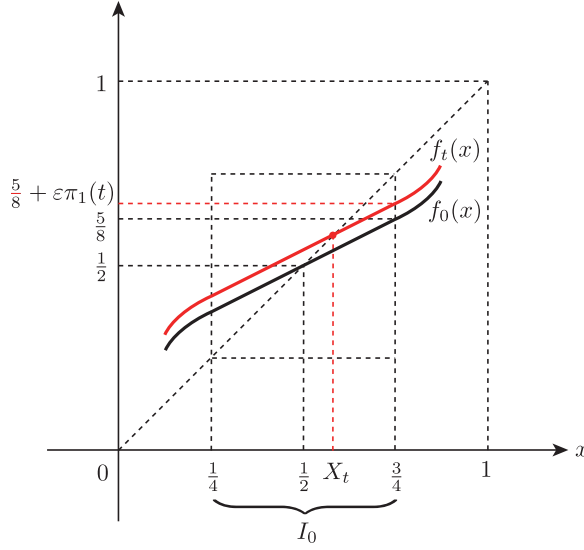
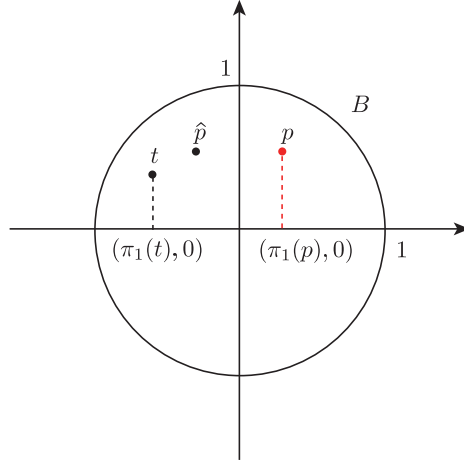
and that $\inf_{x \in \mathbb{S}^1} Df_0(x) > 0$. See Figure 1.

Let B be the two-dimensional closed unit disk, and fix distinct points p and \hat{p} in the interior of B . Let π_1 be the projection from B onto the first axis of B . For simplicity, we assume that the second axis of B passes through the midpoint between p and \hat{p} , and that $\pi_1(p)$ is a positive number. See Figure 2.

Fix $0 < \epsilon < \frac{1}{8}$. We define a parametrised family f of \mathcal{C}^r diffeomorphisms by

$$(2.2) \quad f_t(x) = f_0(x) + \epsilon \pi_1(t) \pmod{1}, \quad (t, x) \in B \times \mathbb{S}^1.$$

³ Note that the $\text{Diff}^r(M, M)$ -valued random variables $\{\bar{t} \mapsto f_{t_n}\}_{n \geq 0}$ on $(B^{\mathbb{N}_0}, \text{Leb}_B^\theta)$ is not necessarily independent nor identically distributed. In particular, $\{\bar{t} \mapsto f_{t_n}\}_{n \geq 0}$ is identically distributed on $(B^{\mathbb{N}_0}, \text{Leb}_B^\theta)$ if and only if Leb_B is invariant under θ : for each $n \geq 0$, if we denote the probability distribution of $\bar{t} \mapsto t_n$ by P_n , then $P_n(E) = \text{Leb}_B(\{t \in B \mid f_{\theta^n t} \in E\}) = \text{Leb}_B(\theta^{-n}\{t \in B \mid f_t \in E\})$ for any Borel set $E \subset \text{Diff}^r(M, M)$. See also Remark 8.

FIGURE 1. Diffeomorphisms f_0 and f_1 .FIGURE 2. Parameter space B and specified points p and \hat{p} .

Note that $f_t(I_0) \subset I_0$ and $f_t|_{I_0} : I_0 \rightarrow I_0$ has a unique fixed point, denoted by X_t , for each $t \in B$. In particular, for $t = p, \hat{p}$,

$$(2.3) \quad X_p = \frac{1}{2} + 2\epsilon\pi_1(p), \quad X_{\hat{p}} = \frac{1}{2} + 2\epsilon\pi_1(\hat{p}) = \frac{1}{2} - 2\epsilon\pi_1(p).$$

We need the following elementary lemma.

Lemma 5. *For any $n \in \mathbb{N}_0$, $x \in I_0$, $\bar{t} = (t_0, t_1, \dots) \in B^{\mathbb{N}_0}$ and $t' \in B$, we have*

$$(2.4) \quad d_{\mathbb{S}^1} \left(f_{\bar{t}}^{(n)}(x), X_{t'} \right) \leq \frac{1}{2^n} + 6\epsilon \max_{0 \leq j \leq n-1} |\pi_1(t') - \pi_1(t_j)|.$$

Proof. When $n = 0$, (2.4) immediately follows from the definition of $d_{\mathbb{S}^1}(\cdot, \cdot)$. Fix $n \geq 1$, $x \in I_0$, $\bar{t} = (t_0, t_1, \dots) \in B^{\mathbb{N}_0}$ and $t' \in B$. Noting that $X_{t'} = \pi_{\mathbb{S}^1}(\frac{1}{2} + 2\epsilon\pi_1(t'))$, we have

$$\begin{aligned} d_{\mathbb{S}^1} \left(f_{\bar{t}}^{(n)}(x), X_{t'} \right) &\leq d_{\mathbb{S}^1} \left(f_{t_{n-1}}(f_{\bar{t}}^{(n-1)}(x)), f_{t_{n-1}}(X_{t_{n-1}}) \right) + d_{\mathbb{S}^1} \left(X_{t_{n-1}}, X_{t'} \right) \\ &\leq \frac{1}{2} d_{\mathbb{S}^1} \left(f_{\bar{t}}^{(n-1)}(x), X_{t_{n-1}} \right) + 2\epsilon |\pi_1(t_{n-1}) - \pi_1(t')|. \end{aligned}$$

Reiterating this argument, we finally get that $d_{\mathbb{S}^1} \left(f_{\bar{t}}^{(n)}(x), X_{t'} \right)$ is bounded by

$$\frac{1}{2^n} d_{\mathbb{S}^1} (x, X_{t_0}) + 2\epsilon \left(\sum_{j=1}^{n-1} \frac{|\pi_1(t_{n-j-1}) - \pi_1(t_{n-j})|}{2^j} + |\pi_1(t_{n-1}) - \pi_1(t')| \right).$$

Hence, the conclusion follows from the triangle inequality

$$|\pi_1(t_{n-j-1}) - \pi_1(t_{n-j})| \leq |\pi_1(t_{n-j-1}) - \pi_1(t')| + |\pi_1(t') - \pi_1(t_{n-j})|.$$

This completes the proof. \square

In the remainder of this subsection, we prove the following lemma.

Lemma 6. *The parametrised family f defined as (2.2) satisfies the absolute continuity condition (1.2) and (1.3) with $M = \mathbb{S}^1$, $N = 1$ and $\xi = \epsilon$.*

Proof. We first see that (1.2) holds, so fix $n \geq 1$ and $x \in \mathbb{S}^1$. Due to that $f_{\bar{t}}^{(n)}(x) = f_{t_{n-1}}(f_{\bar{t}}^{(n-1)}(x))$ for each $\bar{t} \in B^{\mathbb{N}_0}$, $\{f_{\bar{t}}^{(n)}(x) \mid \bar{t} \in B^{\mathbb{N}_0}\}$ contains $\{f_t(f_0^{n-1}(x)) \mid t \in B\}$. Furthermore, by virtue of (2.2), $\{f_t(f_0^{n-1}(x)) \mid t \in B\}$ coincides with the ball with radius ϵ centred at $f_0^{n-1}(x)$. Therefore (1.2) holds with $N = 1$ and $\xi = \epsilon$.

To prove (1.3), we will show that for any $n \geq 1$, there exists a constant $C_n > 0$ such that

$$(2.5) \quad \left(f_{(\cdot)}^{(n)}(x) \right)_* \text{Leb}_B^{\mathbb{N}_0}(A) \leq C_n \text{Leb}_{\mathbb{S}^1}(A)$$

for any $x \in \mathbb{S}^1$ and Borel set $A \subset \mathbb{S}^1$. Arguing by induction, we first see that (2.5) holds for $n = 1$. For each Borel set $\tilde{A}_1 \subset [-1, 1]$, we have

$$\text{Leb}_B \left(\{t \in B \mid \pi_1(t) \in \tilde{A}_1\} \right) \leq 2\text{Leb}_{\mathbb{R}}(\tilde{A}_1)$$

since $\{t \in B \mid \pi_1(t) \in \tilde{A}_1\}$ is included in $\tilde{A}_1 \times [-1, 1]$. Therefore, for any $x \in \mathbb{S}^1$ and Borel set $A \subset \mathbb{S}^1$, if we let \tilde{A} be a Borel set in $[0, 1]$ such that $\pi_{\mathbb{S}^1}(\tilde{A}) = A$, then

$$\left(f_{(\cdot)}^{(1)}(x) \right)_* \text{Leb}_B^{\mathbb{N}_0}(A) = \text{Leb}_B \left(\left\{ t \in B \mid f_0(x) + \epsilon\pi_1(t) \in \tilde{A} \right\} \right) \leq \frac{2}{\epsilon} \text{Leb}_{\mathbb{R}}(\tilde{A}),$$

which coincides with $2/\epsilon \cdot \text{Leb}_{\mathbb{S}^1}(A)$. That is, (2.5) holds with $n = 1$ and $C_1 = 2/\epsilon$.

Suppose that (2.5) holds for $n = k$. Then, for any $x \in \mathbb{S}^1$ and Borel set $A \subset \mathbb{S}^1$, $(f_{(\cdot)}^{(k+1)}(x))_* \text{Leb}_B^{\mathbb{N}_0}(A)$ coincides with

$$\int \text{Leb}_B^{\mathbb{N}_0} \left(\left\{ \bar{t} \in B^{\mathbb{N}_0} \mid f_{\bar{t}}^{(k)}(x) \in f_t^{-1}(A) \right\} \right) d\text{Leb}_B(t),$$

which is bounded by

$$\begin{aligned} \sup_{t \in B} \text{Leb}_B^{\mathbb{N}_0} \left(\left\{ \bar{t} \in B^k \mid f_{\bar{t}}^{(k)}(x) \in f_t^{-1}(A) \right\} \right) &\leq \sup_{t \in B} C_k \text{Leb}_{\mathbb{S}^1}(f_t^{-1}(A)) \\ &\leq C_k \sup_{(t,x) \in B \times \mathbb{S}^1} |Df_t^{-1}(x)| \text{Leb}_{\mathbb{S}^1}(A). \end{aligned}$$

Here we used the inductive step with A replaced by $f_t^{-1}(A)$ in the first inequality. Hence, it follows from (2.2) that (2.5) holds with $n = k + 1$ and $C_{k+1} = C_k \sup_x |Df_0^{-1}(x)|$, and the proof of (2.5) is complete. \square

2.2. Forcing system. In order to provide the required forcing system $\theta : B \rightarrow B$, we borrow results from [6]. Due to conflicting notation, we here restate (and collect) the results in the mentioned article. Let $\text{Diff}^{\tilde{r}}(B, B)$ be the set of all $\mathcal{C}^{\tilde{r}}$ diffeomorphisms on B endowed with the usual $\mathcal{C}^{\tilde{r}}$ metric with $2 \leq \tilde{r} < \infty$.

Theorem 7. *Let $\tilde{\theta} : B \rightarrow B$ be a $\mathcal{C}^{\tilde{r}}$ diffeomorphism in a Newhouse open set of $\text{Diff}^{\tilde{r}}(B, B)$ with $2 \leq \tilde{r} < \infty$. Then any neighbourhood $\mathcal{N}(\tilde{\theta})$ of $\tilde{\theta}$ in $\text{Diff}^{\tilde{r}}(B, B)$ contains an element θ which admits a non-empty open set D in B such that, for any $t \in D$, the forward orbit $\bar{t} = (t, \theta t, \theta^2 t, \dots)$ has a historic behaviour.*

Remark 8. By virtue of Birkhoff's ergodic theorem, Leb_B is not an invariant measure under the nonsingular mapping θ in Theorem 7. Therefore, it follows from the footnote in the paragraph preceding Theorem 3 that $\text{Diff}^r(M, M)$ -valued random variables $\{\bar{t} \mapsto f_{t_n}\}_{n \geq 0}$ on $(B^\theta, \text{Leb}_B^\theta)$ is not identically distributed.

Now we recall the dynamics of θ presented in [6], which is crucial in the proof of our main theorem. The diffeomorphism θ has two saddle fixed points, which are rearranged so that these correspond to the points p and \hat{p} given in Subsection 2.1. We set

$$(2.6) \quad \delta = \min \left\{ \frac{\pi_1(p)}{18}, \frac{\text{dist}_B(p, \partial B)}{2}, \frac{\text{dist}_B(\hat{p}, \partial B)}{2} \right\},$$

and let $U_\delta(p)$ and $U_\delta(\hat{p})$ be the δ -neighbourhoods of p and \hat{p} in B , respectively. Then it follows from (2.6) that $U_\delta(p) \cap U_\delta(\hat{p}) = \emptyset$ and $(U_\delta(p) \cup U_\delta(\hat{p})) \cap \partial B = \emptyset$. In [6], we have actually shown that, for any sufficiently large positive integers z_0 , n_0 , k_0 and any sequence $\mathbf{z} = \{z_k\}_{k=k_0}^\infty$ of integers each entry of which is either z_0 or $z_0 + 1$, there exists an element θ in $\mathcal{N}(\tilde{\theta})$, together with a sequence $\{R_k\}_{k=k_0}^\infty$ of mutually disjoint rectangles in B and sequences $\{a_k\}_{k=k_0}^\infty$, $\{b_k\}_{k=k_0}^\infty$ of positive integers, such that the following conditions hold:

- (C1) $\text{Int}(R_{k_0}) = D$ and $\lim_{k \rightarrow \infty} \text{diam}(R_k) = 0$;
- (C2) $\limsup_{k \rightarrow \infty} a_k/k < \infty$ and $\limsup_{k \rightarrow \infty} b_k/k < \infty$;
- (C3) For any $t \in R_k$ with $k \geq k_0$,
 - $\theta^{a_k + n_0 + j}(t) \in U_\delta(p)$ if $0 \leq j \leq z_k k^2 - 2n_0$,
 - $\theta^{a_k + n_0 + z_k k^2 + j}(t) \in U_\delta(\hat{p})$ if $0 \leq j \leq k^2 - 2n_0$,
 - $\theta^{m_k}(t) \in \text{Int}(R_{k+1})$ for $m_k = (z_k + 1)k^2 + a_k + b_k$.

See Figure 3 for the situation.

For a given monotone increasing sequence $\{k_J\}_{J=1}^\infty$ of integers with $k_1 > k_0$, the sequence $\mathbf{z} = \{z_k\}_{k=k_0}^\infty$ is supposed to satisfy

$$(2.7) \quad z_k = z_0 \quad \text{if } J \text{ is odd} \quad \text{and} \quad z_k = z_0 + 1 \quad \text{if } J \text{ is even}$$

for any $k_{J-1} < k \leq k_J$.

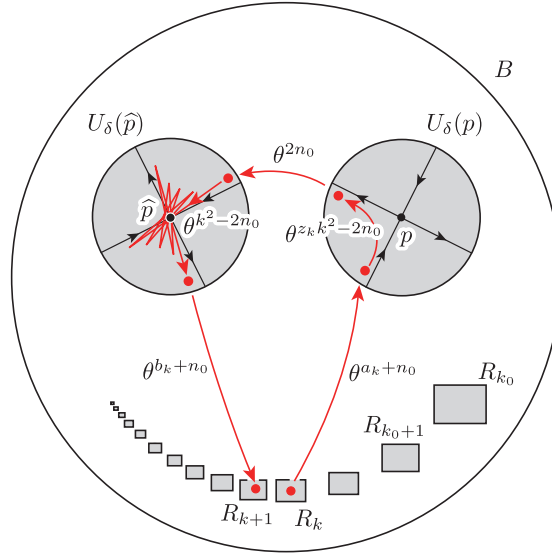


FIGURE 3. Travel from R_k to $\text{Int}(R_{k+1})$ via θ^{m_k} in the case where the eigenvalues of $D\theta(p)$ are positive and those of $D\theta(\hat{p})$ are negative.

2.3. Completion of the proof. Let $x \in I_0$, $\bar{t} = (t, \theta t, \theta^2 t, \dots) \in D^\theta$ and u_0 a nonnegative integer. With the notation \bar{s} for $(\theta^{u_0} t, \theta^{u_0+1} t, \theta^{u_0+2} t, \dots)$, we get

$$f_{\bar{t}}^{(u_0+j)}(x) = f_{\bar{s}}^{(j)}(f_{\bar{t}}^{(u_0)}(x))$$

for any $j \geq 0$. Thus applying (2.4) with n, x, \bar{t} replaced by $j, f_{\bar{t}}^{(u_0)}(x), \bar{s}$, we have

$$(2.8) \quad d_{\mathbb{S}^1}(f_{\bar{t}}^{(u_0+j)}(x), X_{t'}) \leq \frac{1}{2^j} + 6\epsilon \max_{0 \leq i \leq j-1} |\pi_1(t') - \pi_1(\theta^{u_0+i} t)|$$

(in particular, for $t' = p$ or \hat{p}).

Proof of Theorem 3. Since f satisfies the absolute continuity condition by Lemma 6, it suffices to show that, for any $(\bar{t}, x) \in D^\theta \times I_0$, the random orbit $\{f_{\bar{t}}^{(n)}(x)\}_{n \geq 0}$ has historic behaviour.

Let $\mathcal{N}(X_p)$ and $\mathcal{N}(X_{\hat{p}})$ be the $\epsilon\pi_1(p)$ -neighbourhood of X_p and $X_{\hat{p}}$ in \mathbb{S}^1 , respectively, so that $\mathcal{N}(X_p) \cap \mathcal{N}(X_{\hat{p}}) = \emptyset$ by virtue of (2.3). Fix a positive integer n_1 with $2^{-n_1} \leq \epsilon\pi_1(p)/3$. Then it follows from (C3), (2.6) and (2.8) with $u_0 = a_k + n_0$ and $t' = p$ that for any $n_1 \leq j \leq z_k k^2 - 2n_0$,

$$(2.9) \quad d_{\mathbb{S}^1}(f_{\bar{t}}^{(a_k+n_0+j)}(x), X_p) \leq \frac{1}{2^j} + 6\epsilon\delta \leq \frac{\epsilon\pi_1(p)}{3} + \frac{\epsilon\pi_1(p)}{3} = \frac{2}{3}\epsilon\pi_1(p)$$

and hence $f_{\bar{t}}^{(a_k+n_0+j)}(x) \in \mathcal{N}(X_p)$. Similarly we have $f_{\bar{t}}^{(a_k+n_0+z_k k^2+j)}(x) \in \mathcal{N}(X_{\hat{p}})$ for any $n_1 \leq j \leq k^2 - 2n_0$.

Let $\varphi : \mathbb{S}^1 \rightarrow [0, 1] \subset \mathbb{R}$ be a continuous function such that $\varphi(x) = 1$ if x is in $\mathcal{N}(X_p)$ and $\varphi(x) = 0$ if x is in $\mathcal{N}(X_{\hat{p}})$. Write $\hat{m}_k = \sum_{j=k_0}^k m_j$ for any $k \geq k_0$ and the integers m_k given in (C3).

Now we will show that the sequence $\{k_J\}_{J=1}^\infty$ can be taken so that the following inequality holds

$$(2.10) \quad \frac{\sum_{k=k_{J-1}+1}^{k_J} (z_k k^2 - 2n_0 - n_1)}{\hat{m}_{k_J}} \geq \frac{z_*}{z_* + 1} - 2^{-J},$$

where $z_* = z_0$ if J is odd and $z_* = z_0 + 1$ if J is even. Indeed, recall that

$$\frac{\sum_{k=k_{J-1}+1}^{k_J} (z_k k^2 - 2n_0 - n_1)}{\hat{m}_{k_J}} = \frac{\sum_{k=k_{J-1}+1}^{k_J} (z_* k^2 - 2n_0 - n_1)}{\sum_{k=k_{J-1}+1}^{k_J} ((z_* + 1)k^2 + a_k + b_k)} \frac{\hat{m}_{k_J} - \hat{m}_{k_{J-1}}}{\hat{m}_{k_J}}.$$

By taking k_J sufficiently larger than k_{J-1} , one can suppose that $(\hat{m}_{k_J} - \hat{m}_{k_{J-1}})/\hat{m}_{k_J}$ is arbitrarily close to one. Thus it follows from the preceding equation together with (C2) that there exists a sequence $\{k_J\}_{J=1}^\infty$ satisfying (2.10).

In a similar manner, we also can assume that

$$\frac{\sum_{k=k_{J-1}+1}^{k_J} (k^2 - 2n_0 - n_1)}{\hat{m}_{k_J}} \geq \frac{1}{z_* + 1} - 2^{-J}.$$

Then, due to the observation following (2.9), we have

$$\begin{aligned} \frac{1}{\hat{m}_{k_J}} \sum_{j=1}^{\hat{m}_{k_J}} \varphi(f_{\bar{t}}^{(j)}(x)) &\geq \frac{\#\{\hat{m}_{k_{J-1}} < j \leq \hat{m}_{k_J} \mid f_{\bar{t}}^{(j)}(x) \in \mathcal{N}(X_p)\}}{\hat{m}_{k_J}} \\ &\geq \frac{\sum_{k=k_{J-1}+1}^{k_J} (z_k k^2 - 2n_0 - n_1)}{\hat{m}_{k_J}} \geq \frac{z_*}{z_* + 1} - 2^{-J}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\hat{m}_{k_J}} \sum_{j=1}^{\hat{m}_{k_J}} \varphi(f_{\bar{t}}^{(j)}(x)) &\leq 1 - \frac{\#\{\hat{m}_{k_{J-1}} < j \leq \hat{m}_{k_J} \mid f_{\bar{t}}^{(j)}(x) \in \mathcal{N}(X_p)\}}{\hat{m}_{k_J}} \\ &\leq 1 - \frac{\sum_{k=k_{J-1}+1}^{k_J} (k^2 - 2n_0 - n_1)}{\hat{m}_{k_J}} \leq \frac{z_*}{z_* + 1} + 2^{-J}. \end{aligned}$$

Therefore, we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^n \varphi(f_{\bar{t}}^{(j)}(x)) \leq \frac{z_0}{z_0 + 1} < \frac{z_0 + 1}{z_0 + 2} \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^n \varphi(f_{\bar{t}}^{(j)}(x))$$

for all (\bar{t}, x) in $D^\theta \times I_0$. Since D is of Leb_B -positive measure by Theorem 7, D^θ and I_0 has positive measure with respect to Leb_B^θ and $\text{Leb}_{\mathbb{S}^1}$, respectively. This completes the proof of Theorem 3. \square

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